OPTIMAL LOCATIONS OF FACILITIES FOR CONTINUOUS DEMANDS

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Dedicated to

RAKESH

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ABSTRACT

The present work has been undertaken to devise methodologies for determining the optimal locations of new facilities within a region all over which demand for these facilities exists. Models for the location of single new facility and the simultaneous location of multiple new facilities with interaction among the new facilities have been developed.

The thesis, for the most part, centers around the facility location models with continuous demand, but a probabilistic version of the discrete demand single facility location problem has also been developed because this probabilistic version is very much similar to the continuous demand single facility location problem as far as its nature and method of solution are concerned. In this model, the co-ordinates of the destination points are assumed to be random variables instead of deterministic as assumed in the previous works in location theory.

All the above stated models have been solved for three types of distances, viz. rectilinear, squared Euclidean, and Euclidean.

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Chapter 1

INTRODUCTION

Selection of most profitable locations for plants or other facilities has been a subject of analysis for centuries. It has attracted a wide diversity of practitioners including applied mathematicians interested in the mathematical characterization of proposed algorithmic solution models; geographers interested in the spatial patterns of mass human interactions; urban and regional economists attempting to deal with problems of city land use and agricultural location; and planners, systems engineers, and operations researchers concerned with solving specific problems relating to the location of a particular facility. Economists, urban planners, management scientists, architects, regional scientists, home economists, and engineers from several disciplines have discovered a commonality of interest in their concern for the optimal location of facilities.

Even though facility location problems have received considerable interest over the years, it was not until the emergence of the techniques of operations research that the subject was studied in a quantitative manner. In fact, before 1960, the approaches to tackle these problems relied heavily on intuition and engineering judgement, and used only qualitative methods giving no or very less consideration

to the possibility of discovering analytical solution procedures.

The past ten or fifteen years have seen a significant development in what has come to be known as modern location theory to provide the facility analysts with new techniques, approaches and philosophies for the solution of facility location problems and to take a more quantitative approach than was commonly taken before. In fact, there are quantitative aspects of the facility location problems that cannot be reckoned accurately through intuition alone.

Modern location theory provides an in-depth treatment of a relatively limited number of aspects of the locational problems as compared with the traditional approach, which examines a broad range of problems in a less scientific fashion. It deals with a smaller set of problems because it is still in its early stage of development. Many theoretical and applied problems remain to be solved.

Even though heavy emphasis is given, in modern location theory, to analytical approaches, one should not overlook the importance of qualitative aspects of locational problems. It should be realized that the analytical approach yields the solution to the model but not necessarily to the problem. The solutions obtained through analysis serve only as aids in decision making. There remains a number of non-quantifiable factors, which must be considered alongwith

scheme is presented in Chapter 2. It is believed that the proposed classification scheme will provide a modest push to the need of a bird's eye view of the location theory problems.

Although considerable amount of work has been done by previous researchers on the problem of locating the facilities when there are no constraints upon the area in which they can be located, yet the models given by them cannot be used when there are a very large number of demand points needing the services of the new facilities. An attempt has been made in this work to develop mathematical models and solution procedures for continuous demand location problems.

In Chapter 3 we shall describe the problem in which only one new facility is to be located and in Chapter 4 we shall deal with the problem of simultaneous location of n new facilities. In Chapter 5, the probabilistic version of the single facility location problem will be discussed. Our approach will be to first give a brief account of what is already available in literature concerning these models and then to describe in detail the proposed extension of the models.

Chapter 2

CLASSIFICATION OF FACILITY LOCATION PROBLEMS

The development of literature on facility location problems is somewhat haphazard. The available literature on location theory deals with a large number of location problems and there seems to be little co-ordination between them, making it difficult to find where a particular problem stands in the field of location theory. A thorough review of the existing literature on location theory indicates that the location problems can be grouped by using the following scheme :

1. Classification based on the solution space available,
2. Classification based on the optimization criterion,
3. Classification based on the number and characteristics of the facilities to be located,

and 4. Classification based on the planning horizon.

The various types of problems which belong to four classifications listed above are discussed in the following paragraphs.

2.1 Classification Based on the Solution Space Available :

This criterion classifies the facility location problems (FLP's) according to the type of area which is available to us for locating the new facilities. There are three categories of problems which stem from this

classification. The three categories are :

a) Continuous space location problems, b) Discrete space location problems, and c) Location problems on graphs.

a) Continuous Space Location Problems - Facility location problems in which new facilities can be located anywhere in the plane come under this category. The location of small depots, schools, shops and all those facilities which do not require a large floor area or some special commodity which is not available everywhere are some of the typical examples of continuous space location problems.

For locating the facilities in continuous space, one has to determine the distance between various facilities for evaluating the total materials handling cost. Depending on the requirements of the problem, distances are measured in several ways. Most commonly used methods are :

(i) Euclidean, (ii) Rectilinear, and (iii) Squared Euclidean.

(i) Euclidean Distance. The most obvious distance between the facilities is the straight line, or Euclidean, distance. If the co-ordinates of two facilities are (x_1, y_1) and (x_2, y_2) respectively, the Euclidean distance between them is defined as

$$d = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{\frac{1}{2}}$$

Euclidean distance is used for some network location problems as well as some instances involving conveyors

and air travel. Some electrical wiring problems and pipeline design problems are also examples of Euclidean distance problems.

(ii) Rectilinear Distance. In most machine location problems, travel occurs along a set of aisles arranged in a rectangular pattern parallel to the walls of the building. In such a situation, the appropriate distance is variously referred to as the rectilinear, rectangular, or metropolitan distance. We choose the former and define the rectilinear distance between the two machines as

$$d = |x_1 - x_2| + |y_1 - y_2|$$

Rectilinear distance is appropriate in some urban location analyses where travel occurs along an orthogonal set of streets. Additionally, a number of offices employ a rectilinear set of aisles and hallways to facilitate the travel of personnel.

(iii) Squared Euclidean Distance. In some facility location problems cost is not a simple linear function of distance. As an example, the cost associated with the response of a fire station truck to a fire is expected to be non-linear with distance. Depending on the location problem, the cost function can take on a number of different formulations. The most common non-linear form of the cost function is to assume it to be proportional to the square of the Euclidean

distance, or to take the distance itself as

$$d = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

There is another reason also for interest in squared Euclidean distance. The study of the squared Euclidean distance problem lays some of the groundwork for the Euclidean distance problem which is more important but much difficult to solve as compared to the former. The methods of solving Euclidean problems are always of iterative nature and require some initial approximation. It is customary to use the solution of corresponding squared Euclidean problem as this initial approximation. In some cases the solution of squared Euclidean problem can be used as the solution of Euclidean problem also with moderate error (40).

b) Discrete Space Location Problems - The discrete space location problems involve the location of new facilities at some prespecified locations only. Thus, we have to choose the best possible locations for the new facilities from a given set of discrete locations. Problems of locating large warehouses, factories, plants etc. are some of the typical examples belonging to this class. These big facilities cannot be located anywhere in the plane because of the large floor area requirements and/or other factors.

c) Location Problems on Graphs - FLP's in which we can locate new facilities only on the vertices or arcs of some predetermined network are included in this category. Problems

concerning the determination of best locations for facilities in a metropolitan city on its road network, in a region on its railway network or in a telephone system on its communication network, etc., are some of the typical examples. The location problems on graphs are the most practical problems because transportation or communication almost always takes place along some predetermined network.

2.2 Classification Based on Optimization Criterion :

In this classification we group the location problems according to the objective function to be optimized. The two most commonly used optimization criteria are mini-sum and mini-max. Besides these two criteria there are many other special criterion which are instituted based on the nature of the location problem. For further discussion on this classification, location problems have been classified as :

a) Mini-sum location problems, b) Mini-max location problems, and c) special location problems.

a) Mini-sum Location Problems - In this category we place those problems in which cost of total materials handling between new and old facilities and between new facilities themselves is to be minimized. This is an obvious objective function and most of the location problems arising in private sector are solved using this criterion. Examples include location of a warehouse for supplying

commodities to customers, location of a factory to supply goods in various markets and location of a new machine in a machine shop, etc.

b) Mini-max Location Problems - In this class of problems the minimization of the maximum of all distances (or costs, travel times, etc.) from the new facility to various old facilities is used as the criterion of optimization. The problems of locating a hospital in rural area, the placement of firestations in a city, the determination of the best location of a police station in a town are handled using the mini-max objective. In these cases it is not fair to minimize the total man-miles travelled per unit of time, because a minimum man-mile solution may result in a situation in which some people, may be very few, have to travel a great distance to reach to the service centre and when some emergency arises, they cannot be served in time. In this case, therefore, we should minimize the maximum distance any person is required to travel to reach the nearest service centre. Hence mini-max criterion must be used for such problems. In general, mini-max criterion is used in the location of public sector facilities, specially those which are required at emergency.

Another justification for minimizing maximum rather than the total distance in the location of public sector facilities is that one who installs such facilities does

not pay the transportation costs unlike to the location of plants and warehouses etc. where all materials handling costs are to be paid by the owner of the warehouse. In the case of public facilities materials handling costs are paid by those who use these facilities and hence we should minimize the maximum of the individual costs.

c) Special Location Problems - In addition to the two optimization criteria discussed above, there are some location problems in which we are forced to optimize using some need based criterion. For example, we sometimes come across the problems of network design in which several points on plane are given, and we are required to join these points with smallest possible total link length. These points can be joined in two ways. One is without creating any new node which means that junctions are only at given points, and the other is when this restriction is released, i.e., we can create new junctions also. When the second method, which is more realistic, is used we have a location problem in which we will have to determine the optimum number of extra junction points, their locations and then the optimum linking between various points so that the total link length is minimized. These problems arise in real life when we have to join different towns by minimum possible transportation network or different points in a water supply system by minimum length of pipe line etc.

Another special location problem arises while determining the locations of emergency service facilities. For such problems it is desired to locate the facilities such that no requirement centre is farther away than a prespecified critical distance or critical travel time from at least one facility, i.e., all the requirement centres are "covered". Thus we want to determine the minimum number of facilities and their locations so that the objective is accomplished (total covering problem), or given that a particular number of facilities are to be located, to determine their locations such that maximum number of requirement centres can be covered (partial covering problem).

2.3 Classification Based on the Number and Characteristics of the Facilities to be Located :

Under this classification, the location problems can be grouped into the following classes :

a) Single facility location problems, in which only one new facility is to be located. This problem when formulated in continuous space with a mini-sum objective is called the "Depot Location Problem". The depot location problem with Euclidean distances is called the "Steiner-Weber Problem", "Weber Problem", or the "General Fermat Problem".

b) Multi-facility location problems, in which we consider the location of two or more new facilities. Essentially

there are two types of multi-facility location problems :

(i) Problems of locating dissimilar facilities, i.e. those facilities which serve different purposes (called DSFLP's henceforth).

(ii) Problems of locating similar facilities, i.e., those facilities which serve the same purpose (called SFLP's henceforth). In SFLP's we have to determine optimal allocation also, i.e., the allocation of new facilities to serve the various demand centres.

DSFLP's can again be divided according to how the old and new facilities are interactiong, i.e., what is the pattern of materials handling between them. Using this criterion, we can subdivide these problems into three categories:

A. When new facilities interact with old ones only.

B. When new facilities interact within themselves only.

C. When new facilities interact with old ones as well as within themselves.

It seems, at first sight, that the new facilities in any DSFLP must have interaction with the old facilities as well as within themselves, i.e., they must belong to the third category only because if there is no interaction between the new facilities then a multifacility location problem in which n new facilities are to be located can be converted into n single facility location problems. On the other hand, if there is no interaction between the new and old facilities then there is no location problem (all the new facilities

must be located at the same place and this place may be anywhere in the plane !). Hence only the problems of the third category seem to be meaningful and this classification seems to be needless. But when we are locating the facilities in discrete space or on networks, then almost always we have a restriction that no more than one facility can be located at one given site. Under such circumstances, an indirect type of interaction develops between the new facilities. This validates the above classification. Of course, location problems in continuous space need not be classified in this manner. They always belong to the third category only. In discrete space, the problems of first and second category are referred to as the "Assignment Problem" and "Assignment of Facilities to Locations (or the Quadratic Assignment) Problem" respectively. The problem of third category is yet to be posed and solved.

SFLP's can be further divided in two ways. Following two types of problems emerge out from the first of them :

A. Problems in which the number of new facilities to be located is known. Such problems arise in the location of public sector facilities where the number of new facilities is limited due to financial constraints.

B. Problems in which the number of new facilities to be located is unknown. These are problems arising in private sector where the optimal number of new facilities is to be determined in addition to their locations. It

seems, at first sight, that in this case also we shall have restriction on the number of new facilities due to financial limitations and hence it must be known. But here the difference is that if we decide to locate too less number of new facilities then the saving due to this will reflect in the increased transportation costs since all the costs are charged to the owner of the facilities. Hence in such situations we have to determine the optimum number of new facilities such that the overall cost of installing the new facilities and transportation is minimum. This problem, when formulated in continuous space with Euclidean distances, is called "Location-Allocation Problem" and in discrete space, the "Plant Location Problem" or the "Warehouse Location Problem".

The other way of dividing SFLP's gives following two types of problems :

A. Uncapacitated SFLP's, in which there is no upper limit on the capacity of the new facilities to be located.

B. Capacitated SFLP's, in which there is a restriction on the capacity of the new facilities, so that not more than a given amount can be supplied from any new facility.

No capacitated SFLP was formulated in continuous space till very recently. In 1972, Cooper (7) formulated and solved the problem and called it "The Transportation-Location Problem", which actually is the capacitated SFLP

in continuous space or the "Capacitated Location-Allocation Problem".

SFLP's belonging to the above two types in discrete space are called "Uncapacitated Plant (or Warehouse) Location Problems" and "Capacitated Plant (or Warehouse) Location Problems" respectively, when the optimal number of new facilities also is to be determined.

Fig. 2.1 depicts the classification on the basis of number and characteristics of new facilities in a concise format.

2.4 Classification Based on the Planning Horizon :

We can also classify the location problems as to whether it is

a) a static location problem

or b) a dynamic location problem.

In the first category we keep those problems in which the input data, i.e. the parameters of the problem (the number and location of demand points, their level of demands, etc.) belong to only one point in time. This implies that the planning is done as if the parameters of the problem are not variable with time. The second category includes those problems for which parameters are functions of time within the planning period.

Every effort has been made to envelope the whole range of the literature presently available on modern location theory in the proposed classification scheme. However,

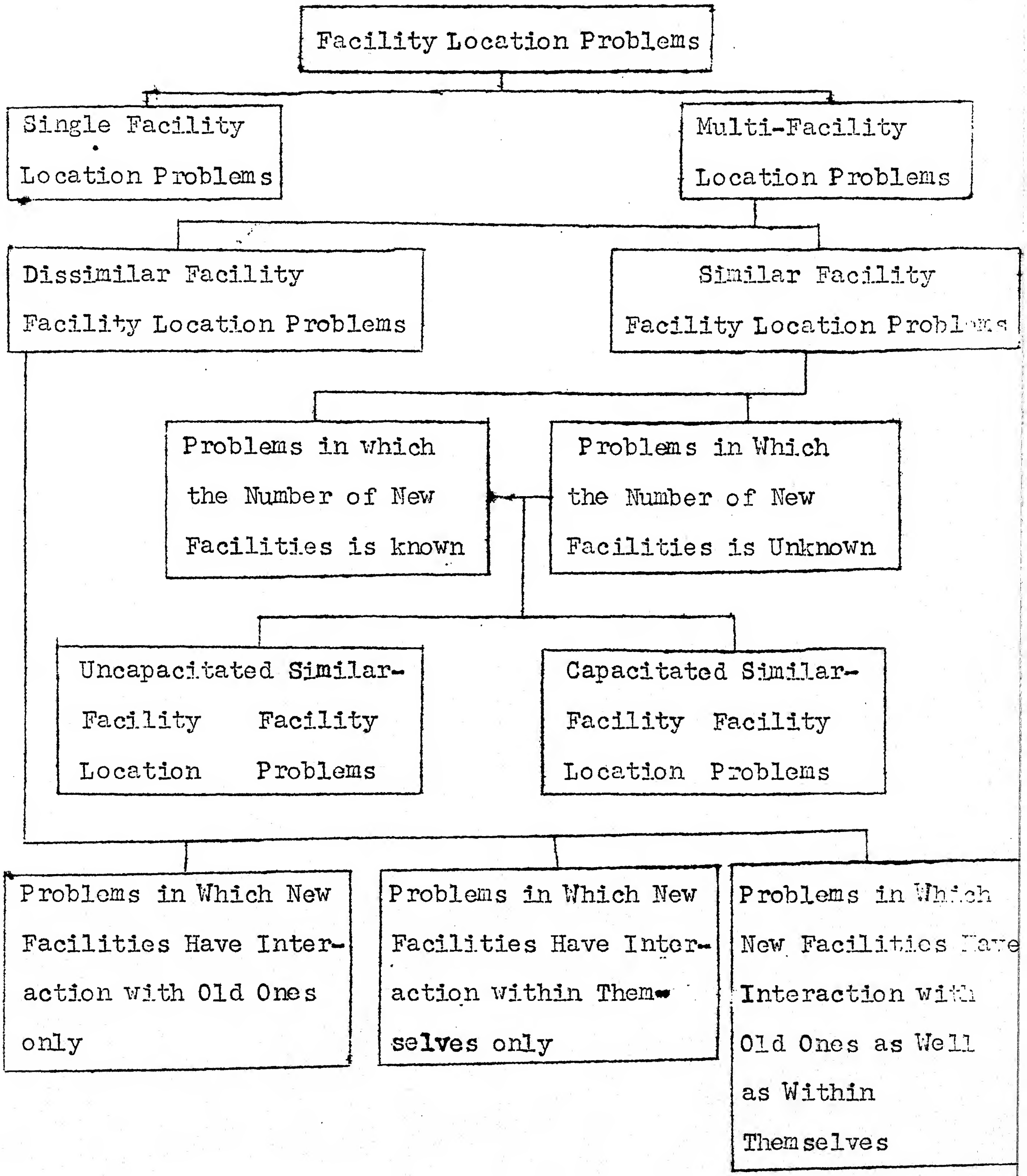

Fig. 2.1 : Classification of Facility Location Problems on the basis of number and characteristics of new facilities.

the literature on the location problems is expanding at a very fast pace and it is becoming increasingly difficult to keep a complete tract on them. Therefore, there may be some location problems which have not been included in any of the categories suggested above. It is not to say, therefore, that these classifications are complete in themselves. Still they give a broad idea of the entire domain of modern location theory and help to overview it.

Chapter 3

SINGLE FACILITY LOCATION PROBLEMS

The objective in single facility location studies is to determine, for the new facility, an optimal location which will minimize appropriately defined total materials-handling cost or total distance function. The cost of materials handling is considered to be proportional to the distance.

There are a number of real life situations which involve the location of a single new facility in an existing layout. Some typical examples of single facility location problems are the location of a new lathe in a manufacturing job shop; a new warehouse relative to production facilities and customers; hospital, firestation, police station, or library in a metropolitan area or a new power generating plant etc.

3.1 Single Facility Location Models :

In literature, the problem of single facility location is posed as follows : n existing facilities are located at known distinct points $P_1,\ldots,P_m$; a new facility is to be located at a point X; costs of transportation are incurred that are directly proportional to an appropriately determined distance between the new facility and existing facilities. Let $d(X, P_i)$ represent the distance travelled

per trip between points X and P_i. If w_i represent the number of trips made per year between the new facility and existing facility i, the total annual cost due to travel between the new facility and all existing facilities is given by

$$f(X) = \sum_{i=1}^{m} w_i \, d(X, P_i) \tag{3.1}$$

The w_i terms are referred to as "weights". The single facility location problem is to determine the location of the new facility that minimizes f(X), the total annual transportation cost.

Depending upon the type of distance function $d(X, P_i)$ used in a particular problem, the single facility location problem is classified as :

1. Rectilinear distance location problem,
2. Squared Euclidean distance location problem,

and 3. Euclidean distance location problem.

The formulations and solution procedures for these problems are presented in the following sections.

<u>3.1.1 Rectilinear Distance Location Problem</u> - The rectilinear distance location problem can be stated mathematically as given below. If the co-ordinates of the new facility are (x, y) and of the existing facility i are (a_i, b_i) so that $X = (x, y)$ and $P_i = (a_i, b_i)$, the location problem is given by

$$\underset{x,y}{\text{Min}} \quad f(x, y) = \sum_{i=1}^{m} w_i \left(|x-a_i| + |y-b_i| \right) \qquad (3.2)$$

The problem given by (3.2) was first solved by Francis (10) in 1963. He gave a simple procedure to find the optimum values of x and y co-ordinates of the new facility and discussed some interesting properties of the problem. He has also given an algorithmic procedure to construct the contour lines[1] for the problem. His solution procedure is based on the duality theory of linear programming.

<u>3.1.2 Squared Euclidean Distance Location Problem</u> - The single facility squared Euclidean distance location problem can be stated as follows :

$$\underset{x,y}{\text{Min}} \quad f(x,y) = \sum_{i=1}^{m} w_i \left[(x-a_i)^2 + (y-b_i)^2 \right] \qquad (3.3)$$

An interest in the squared Euclidean distance problem or the gravity problem (another name of this problem whose justification will be given in a minute) is not new. According to Kelly (20), interest in this problem dates back atleast to the work of Lagrange. McHose (29) has used the basic concepts of differential calculus to obtain the optimal solution for the gravity problem.

---

1 A contour line is a line of constant cost in the plane. Locating the new facility at any point on a contour line results in the same total cost.

$$\min_{x,y} f(x, y) = \sum_{i=1}^{m} w_i (|x-a_i| + |y-b_i|) \qquad (3.2)$$

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Recently Cooper (4,5,6), Kuhn and Kuenne (21), Michle (30), Weiszfeld (41), Vergin and Rogers (40) have used the solution to the single facility gravity problem to initiate a search for the optimum solution to the single facility Euclidean distance problem or the generalized Fermat problem(21).

The gravity problem has a simple solution. The optimal co-ordinates of the new facility are the weighted averages of the corresponding co-ordinates of the existing facilities or in other words the centre of gravity solution is optimal to this problem. Hence, the title gravity problem.

Contour lines for this problem are concentric circles centered on the optimum location of the new facility, i.e., the centroid of the existing facilities' locations.

3.1.3 Euclidean Distance Location Problem. The Euclidean problem may be stated as :

$$\underset{x,y}{\text{Min}} \quad f(x,y) = \sum_{i=1}^{m} w_i [(x-a_i)^2 + (y-b_i)^2]^{1/2} \qquad (3.4)$$

A version of the Euclidean problem for the case $m = 3$, $w_i = 1$, for $i = 1,2,3$, was posed, purely as a problem in geometry, by Fermat early in the seventeenth century, and was solved by Toricelli prior to 1640. The problem was studied by Steiner, a Swiss mathematician, in the nineteenth century, and by Weber, a German economist, early in the twentieth century. Interestingly enough,

however, it was not until the work by Kuhn (21), in 1962, that the problem could be considered to be essentially completely solved.

The Euclidean problem can either be solved by the modified gradient iterative procedure of Kuhn (23) which is guaranteed to converge to the optimal location; a discussion of convergence for this is given by Katz (19), Kuhn (22), and Weiszfeld (41); or by the Hyperboloid Approximation Procedure (HAP) given by Eyster, White, and Wierwille (8). HAP is an iterative procedure based on the use of an approximating function and can be used to solve the rectilinear distance location problem as well.

Unfortunately, exact methods for constructing contour lines are not available for the Euclidean problem, except for the simplest cases where there are only one or two existing facilities. For other cases, a relatively simple method is to obtain approximate contour lines by evaluating the cost function over, say, a rectangular grid of points covering the ranges of (x,y) values of interest. The contour lines can then be drawn by interpolating between grid points. Alternatively, one can assign a given value k to $f(x,y)$ in (3.4), pick a value of x, and search over y for the two values that yield the value k. The process is continued for successive values of x until a family of points is obtained for the contour line having value k. By repeating the same procedure for different

values of k many contour lines can be drawn.

3.2 Proposed Extension of the Models :

In certain classes of location problems, the number of existing facilities or demand points may be excessively large. This would cause high computational effort for the optimal location of the new facility to obtain. Further the representation of large number of demand points or facilities becomes inconvenient. For example, if a problem involves the location of a new facility (distribution point) in a densely populated urban area, it would be most difficult to represent the demand points (each resident) with a point.

A common method for solving such problems is to divide the total area into sub-areas and to represent the population of each sub-area as an aggregate "point", i.e., the centre of gravity of that sub-area. A shortcoming of this method is the error introduced by the assumption that the distance travelled from any point in a sub-area to the new facility equals the distance travelled from the centre of that sub-area to the new facility. Two models which are more accurate representation of the actual situation are proposed here.

In the first model, we divide the locations into groups, each group having some average population density. The second model utilizes a demand density function defined all over the region under consideration. We shall

describe the details of these models one by one.

3.3 Model I : Rectangular Area Approach :

In this model the total region under consideration is divided into rectangularly shaped areas of known dimensions. Each area is assumed to have a uniformly distributed population density, i.e., the demand density at each point in a particular area is same. The demand density may be different for different rectangular areas. Naturally, the accuracy of representation increases with higher number of rectangles used for dividing the total region.

The idea of using uniformly distributed populations for individual rectangles is not a new one. Several contributions to the facilities location literature that use this concept have appeared. Leamer (25) used heuristic methods and straight line distances to allocate sources to the interior of an area with uniformly distributed densities. Francis (13,16) has described several models with continuous destinations. Bender and Goldman (1) and Witzgall (47) have published models of post office systems where the districts have uniformly distributed populations. Newman(32) has proposed a parking lot location problem and has given a solution for the special case where the facilities are straight line segments. A solution method for the case where distances are given by the generalized l_p

function[1] has not yet been developed (36).

In order to formulate the total travel cost, consider a typical rectangle with length $b_i - a_i$ and width $d_i - c_i$. Let X_1 be the location of the new facility in the region under investigation. If m_i represents the density for the ith rectangle, then the travel cost for this rectangle is given by

$$f_i(X_1) = m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} d(X_1, X)\, dy\, dx$$

where $X = (x, y)$ is any point within the rectangle.

Let n be the total number of rectangles. Then the total travel cost for all the rectangles will be

$$f(X_1) = \sum_{i=1}^{n} f_i(X_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} d(X_1,X)\, dy\, dx \quad (3.5)$$

which is to be minimized with respect to the location of the new facility, i.e., $X_1 = (x_1, y_1)$.

We shall solve this problem for all the three types of distances discussed earlier.

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1 If we are given two points in plane, $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$, then the l_p distance between X_1 and X_2 is given by

$$l_p(X_1,X_2) = [\,|x_1 - x_2|^p + |y_1 - y_2|^p\,]^{1/p}$$

Rectilinear and Euclidean distances are two special cases of l_p distance with p=1 and 2 respectively.

3.3.1 Rectilinear Distance Location Problem - When rectilinear distances are used, the location problem becomes

$$\underset{x_1,y_1}{\text{Min}} \quad f(x_1,y_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [|x-x_1| + |y-y_1|] \, dy \, dx \qquad (3.6)$$

Fortunately, (3.6) is separable and the optimum values of x_1 and y_1 can be found independently. This follows because

$$f(x_1,y_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} |x - x_1| \, dydx + \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} |y-y_1| \, dydx$$

$$= \sum_{i=1}^{n} m_i (d_i-c_i) \int_{a_i}^{b_i} |x-x_1| dx + \sum_{i=1}^{n} m_i(b_i-a_i) \int_{c_i}^{d_i} |y-y_1| dy$$

$$= \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} |x-x_1| \, dx + \sum_{i=1}^{n} B_i \int_{c_i}^{d_i} |y-y_1| \, dy$$

$$= f_1(x_1) + f_2(y_1) \qquad \text{(say)}$$

where $A_i = m_i (d_i-c_i)$ and $B_i = m_i (b_i-a_i)$.

The optimum x_1 and y_1 values can therefore be found out by considering the location of facilities among population lines along x and y axes repsectively. We shall discuss the procedure for minimizing $f_1(x_1)$ only because

the minimization of $f_2(y)$ can be done by following the same procedure.

The minimization of $f_1(x_1)$ can be thought of as a one dimensional location problem. In this problem we have to determine the optimal location of a single facility with respect to line destinations placed on an axis. These line destinations can overlap each other. It is convenient to pose an equivalent problem where none of the destinations overlap. For example, if there are two destinations $[a_i, b_i]$ and $[a_j, b_j]$, such that $a_i < a_j$, $a_j < b_i$ and $b_j > b_i$, then

$$A_i \int_{a_i}^{b_i} |x-x_1| \, dx + A_j \int_{a_j}^{b_j} |x-x_1| \, dx$$

$$= A_i \int_{a_i}^{a_j} |x-x_1| \, dx + (A_i+A_j) \int_{a_j}^{b_i} |x-x_1| \, dx + A_j \int_{b_i}^{b_j} |x-x_1| \, dx$$

and three non-overlapping lines $[a_i, a_j]$, $[a_j, b_i]$, and $[b_i, b_j]$ are obtained. After the overlaps have been identified, there will be n' line destinations where $n' \geq n$. We can arrange the destinations along the axis from left to right and label them $[r_i, s_i]$, $i=1,\ldots,n'$, where $r_{i+1} \geq s_i$, $r_i < s_i$, $a_i' = r_i$, $b_i' = s_i$ and the constant is A_i'. With these modifications our objective function will look something like

$$f_1(x_1) = \sum_{i=1}^{n'} A_i' \int_{a_i'}^{b_i'} |x-x_1| \, dx$$

All the terms in $f_1(x_1)$ are convex, and hence $f_1(x_1)$ is also convex. A term such as $A_i' \int_{a_i'}^{b_i'} |x-x_1|\,dx$ and its derivative is plotted in Fig.3.1

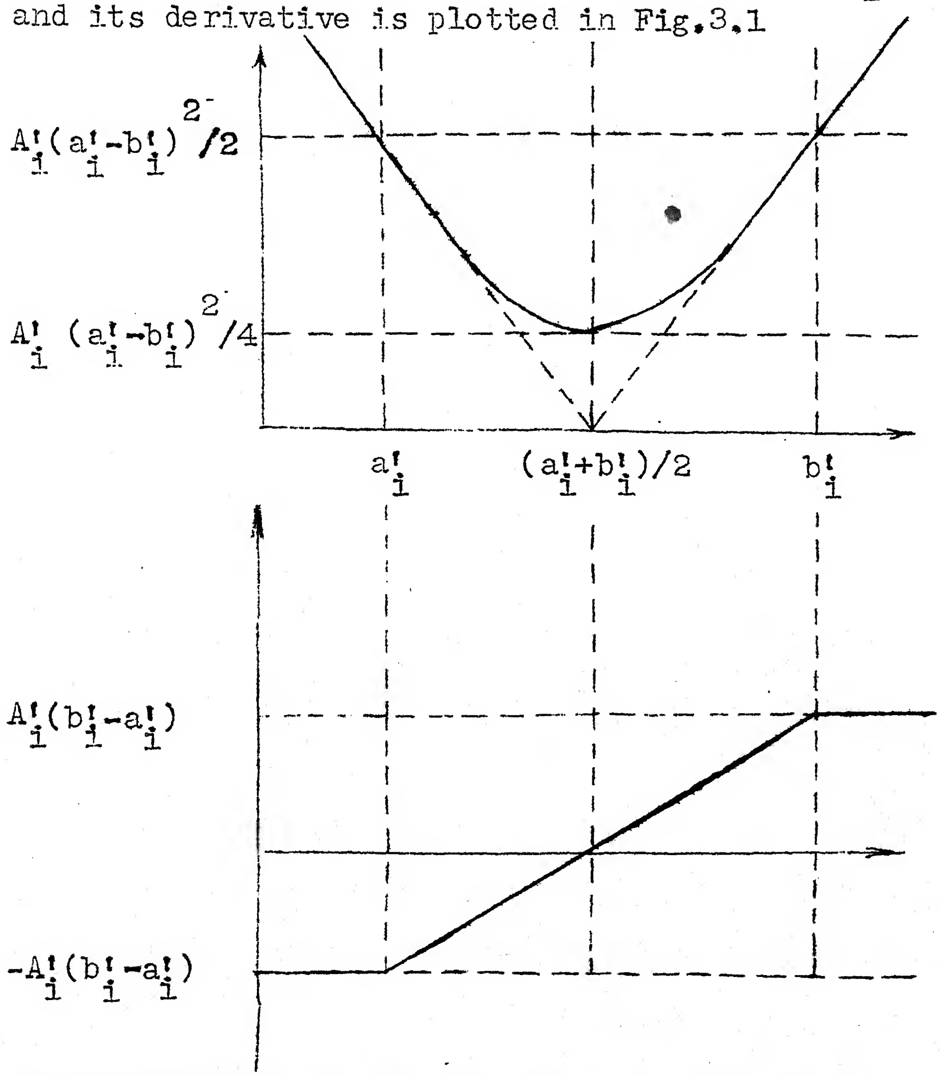

Fig.3.1 : Plot of a typical term in $f_1(x_1)$ and the derivative of that term.

From the figure it is evident that when $x_1 < a_i'$, the slope of the term corresponding to $[r_i, s_i]$ is a negative constant and when $x_1 > b_i'$ the slope is positive value of that same constant. It is possible to evaluate $d\, f_1(x_1)/dx_1$

at the values s_i^+, $i = 1,\ldots,n'$.

Let the absolute value of the constant slope corresponding to $[r_i, s_i]$ be t_i, $i = 1,\ldots,n'$. Then $t_i = A_i' \, (b_i'-a_i')$. When $x_1 < r_1$

$$d\, f_1(x_1)/dx_1 = - \sum_{i=1}^{n'} t_i = - M \text{ (say)}$$

Consequently

$$d\, f_1(x_1)/dx_1 \Big|_{s_i^+} = - M + 2 \sum_{j=1}^{i} t_j \tag{3.7}$$

Since $f_1(x_1)$ is convex, this expression enables us to find the region in which $f_1(x_1)$ is a minimum. For this purpose (3.7) is evaluated for successively larger intervals i, until it becomes either zero or positive. If it becomes zero for some value of i, say i', then $s_{i'} \leq x_1^* \leq r_{i'+1}$, where x_1^* is the optimum value of x_1. If (3.7) becomes positive for the first time when $i = i'$, then $r_{i'} \leq x_1^* \leq s_{i'}$. In the latter case, the exact position of x_1^* can be found out by using the derivative plotted in Fig. 3.1, whereas in the former case x_1^* can be taken any where in the interval.

The co-ordinate y_1^* of the optimum location of the new facility can be found in a similar manner.

We have previously discussed the importance of contour lines while dealing with the discrete demand facility location problems. The contour lines have the same importance in continuous demand location problems also

and now method will be described to construct them for continuous demand location problem.

In this case the equation of the contour line will be

$$k = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} |x - x_1| + |y - y_1| \; dy \; dx$$

where k is a constant.

The above equation can be rewritten as

$$k = \sum_{i=1}^{n} m_i (d_i - c_i) \int_{a_i}^{b_i} | x - x_1 | dx + \sum_{i=1}^{n} m_i (b_i - a_i) \int_{c_i}^{d_i} |y - y_1| dy$$

$$= \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} |x - x_1| \; dx + \sum_{i=1}^{n} B_i \int_{c_i}^{d_i} |y - y_1| \, dy$$

Now, suppose that the point (x_1, y_1) is such that $a_i \leq x_1 \leq b_i$ and $c_i \leq y_1 \leq d_i$. Then we shall have

$$k = \sum_{i=1}^{n} A_i \left[\int_{a_i}^{x_1} (x_1 - x) \; dx + \int_{x_1}^{b_i} (x - x_1) dx \right] + \sum_{i=1}^{n} B_i \left[\int_{c_i}^{y_1} (y_1 - y) dy \right.$$

$$\left. + \int_{y_1}^{d_i} (y - y_1) dy \right]$$

Rearranging the above equation, we obtain

$$k = \left(\sum_{i=1}^{n} A_i\right)x_1^2 + \left(\sum_{i=1}^{n} B_i\right)y_1^2 - \left[\sum_{i=1}^{n} (a_i+b_i)A_i\right] x_1 -$$

$$\left[\sum_{i=1}^{n} (c_i+d_i)B_i\right] y_1 + \tfrac{1}{2} \sum_{i=1}^{n} \left[A_i(a_i^2+b_i^2)+B_i(c_i^2 + d_i^2)\right]$$

This is a second degree equation between x_1 and y_1 and is the required contour line for the rectilinear problem. This equation is valid for the rectangle which extends to $[a_i, b_i]$ in x-direction and to $[c_i, d_i]$ in y-direction. One such equation is to be determined for each of the rectangles to construct one contour line. By taking different values of k, many contour lines may be constructed.

3.3.2 Squared Euclidean Distance Location Problem - The continuous demand squared Euclidean distance location problem or the continuous demand gravity problem will be given by

$$\underset{x_1, y_1}{\text{Min}} \quad f(x_1, y_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} \left[(x-x_1)^2 + (y-y_1)^2\right] dy\, dx \tag{3.8}$$

It is observed that (3.8) is separable in x_1 and y_1 and we can minimize it by breaking it into two components, as shown below.

The component corresponding to x_1 is

$$f_1(x_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} (x-x_1)^2\, dy\, dx$$

$$= \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} (x-x_1)^2 \, dx$$

Differentiation with respect to x_1 gives

$$df_1(x_1)/dx_1 = \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} [\, d(x-x_1)^2/dx_1 \,] \, dx$$

$$= \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} -2\,(x-x_1) \, dx$$

For obtaining a minimum value of $f_1(x_1)$, we must equate $df_1(x_1)/dx_1$ to zero, getting

$$\sum_{i=1}^{n} A_i \int_{a_i}^{b_i} -2\,(x-x_1) \, dx = 0$$

Solving the above equation for x_1, we have

$$x_1 = [\, \sum_{i=1}^{n} A_i\,(b_i^2 - a_i^2) \,] / [\, 2 \sum_{i=1}^{n} A_i (b_i - a_i) \,]$$

Similarly optimum value of y co-ordinate can be determined.

Contour Lines. In this case the equation of contour line becomes

$$k = \sum_{i=1}^{n} A_i \int_{a_i}^{b_i} (x-x_1)^2 dx + \sum_{i=1}^{n} B_i \int_{c_i}^{d_i} (y-y_1)^2 \, dy$$

Evaluating the integrals and rearranging, we obtain

$$k = \left[\sum_{i=1}^{n} A_i(b_i - a_i)\right] x_1^2 + \left[\sum_{i=1}^{n} B_i(d_i - c_i)\right] y_1^2 -$$

$$\left[\sum_{i=1}^{n} A_i(b_i^2 - a_i^2)\right] x_1 - \left[\sum_{i=1}^{n} B_i(d_i^2 - c_i^2)\right] y_1 +$$

$$+ \frac{1}{3} \sum_{i=1}^{n} \left[A_i(b_i^3 - a_i^3) + B_i(d_i^3 - c_i^3)\right]$$

which is the required equation of the contour line.

<u>3.3.3 Euclidean Distance Location Problem</u> - In this case the problem will be given by

$$\operatorname*{Min}_{x_1, y_1} f(x_1, y_1) = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} \left[(x - x_1)^2 + (y - y_1)^2\right]^{\frac{1}{2}} dy dx \tag{3.9}$$

For a minimum value of $f(x_1, y_1)$, we must equate the partial derivatives of $f_1(x_1, y_1)$ with respect to x_1 and y_1 to zero, obtaining

$$\partial f / \partial y_1 = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} \frac{(x - x_1)}{\left[(x - x_1)^2 + (y - y_1)^2\right]^{\frac{1}{2}}} dy dx = 0 \tag{3.10}$$

and

$$\partial f / \partial y_1 = \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} \frac{(y - y_1)}{\left[(x - x_1)^2 + (y - y_1)^2\right]^{\frac{1}{2}}} dy dx = 0 \tag{3.11}$$

Let $[(x-x_1)^2 + (y-y_1)^2]^{\frac{1}{2}} = d$, then (3.10) and (3.11) after rearrangement give the following iterative formulae :

$$x_1^{(h+1)} = \frac{\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [x/d^{(h)}] \, dydx}{\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}] \, dydx} \qquad (3.12)$$

and

$$y_1^{(h+1)} = \frac{\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [y/d^{(h)}] \, dydx}{\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}] \, dydx} \qquad (3.13)$$

The superscripts in the above formulae indicate the iteration number.

The initial values of x_1 and y_1 in (3.12) and (3.13) can be taken as those optimal for a corresponding squared Euclidean problem. The stopping criterion may be based on successive changes in the x and y co-ordinates of the new facility or on successive change in the value of objective function or on both. Since our objective function is convex the iterative procedure is convergent.

The minimization of (3.9) can be done by non-linear programming also and it is interesting to compare (3.12) and (3.13) with the iterative procedure used when

this problem is solved as an unconstrained non-linear programming problem using the steepest descent method.

From (3.10) we get

$$\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [x/d^{(h)}]\, dydx = x_1^{(h)} \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}]\, dydx - \partial f/\partial x_1^{(h)} \tag{3.14}$$

Substituting (3.14) in (3.12) and solving for x_1, we obtain

$$x_1^{(h+1)} = \frac{x_1^{(h)} \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}]\, dydx - \partial f/\partial x_1^{(h)}}{\sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}]\, dydx}$$

or $$x_1^{(h+1)} = x_1^{(h)} - t^{(h)}\, \partial f/\partial x_1^{(h)} \tag{3.15}$$

where $$t^{(h)} = \left\{ \sum_{i=1}^{n} m_i \int_{a_i}^{b_i} \int_{c_i}^{d_i} [1/d^{(h)}]\, dy\, dx \right\}^{-1} \tag{3.16}$$

Similarly

$$y_1^{(h+1)} = y_1^{(h)} - t^{(h)}\, \partial f/\partial y_1^{(h)} \tag{3.17}$$

It is evident that our procedure differs from the steepest descent method since the value of $t^{(h)}$ given by (3.16) need not be an optimal step length. However, the ease with which the value of $t^{(h)}$ is calculated using (3.16)

results in a solution procedure which is easier than the steepest descent method. Furthermore, no background in non-linear programming is required to apply our procedure.

Contour Lines. Similar to Euclidean distance discrete facility location problem, in this case also no direct method is available to construct the contour lines. They are to be drawn as discussed previously for the discrete demand problem.

3.4 Model II : Density Function Approach :

The second model to be described defines a demand density function $g(x,y)$ all over the region under consideration. This means that the demand at an infinitesimal area $dxdy$ at point (x,y) is $g(x,y)dxdy$. We shall solve the single facility location problem using this approach for all the three distances previously discussed. It will be assumed that the region under consideration extends to a,b in x-direction and to c,d in y-direction

3.4.1 Rectilinear Distance Location Problem - The total cost of transportation in this case, will be given by

$$f(x_1,y_1) = \int_a^b \int_c^d g(x,y)\,[\,|x-x_1| + |y-y_1|\,]\,dydx$$

$$= \int_a^b \int_c^d g(x,y)\,|x-x_1|\,dydx + \int_a^b \int_c^d g(x,y)\,|y-y_1|\,dydx$$

results in a solution procedure which is easier than the steepest descent method. Furthermore, no background in non-linear programming is required to apply our procedure.

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$$= \int_a^b \int_c^d g(x,y)\,|x-x_1|\,dydx + \int_a^b \int_c^d g(x,y)\,|y-y_1|\,dydx$$

$$= \int_a^b |x-x_1| \int_c^d g(x,y)\, dydx + \int_c^d |y-y_1| \int_a^b g(x,y)\,dxdy$$

$$= \int_a^b |x-x_1|\, F(x)\, dx + \int_c^d |y-y_1|\, G(y)\, dy$$

where $F(x)$ and $G(y)$ are the marginal density functions of the density function $g(x,y)$ with respect to x and y respectively.

Since the two terms in the expression for $f(x,y)$ are independent of each other we can minimize them separately.

Let $f(x_1,y_1) = f_1(x_1) + f_2(y_1)$,

then $f_1(x_1) = \int_a^{x_1} (x_1-x)\, F(x)\, dx + \int_{x_1}^b (x-x_1)\, F(x)\, dx$

$$= \int_a^{x_1} x_1\, F(x)\, dx - \int_a^{x_1} xF(x)\, dx + \int_{x_1}^b F(x)\,dx - \int_{x_1}^b x_1\, F(x)\, dx$$

Differentiating with respect to x_1 we get

$$d\, f_1(x_1)/dx_1 = x_1F(x_1)+x_1F(x_1)+ \int_{x_1}^b F(x)\,dx - \int_a^{x_1} F(x)\,dx + x_1[\,-F(x_1)-F(x_1)\,]$$

$$= \int_{x_1}^{b} F(x)\,dx - \int_{a}^{x_1} F(x)\,dx$$

For minimum value of $f_1(x_1)$, this must be equal to zero, so that we get the condition of optimality as

$$\int_{a}^{x_1} F(x)\,dx = \int_{x_1}^{b} F(x)\,dx$$

Since $\int_{a}^{b} F(x)\,dx = 1$, both the two sides of the above equation must be equal to 1/2, i.e.

$$\int_{a}^{x_1} F(x)\,dx = \int_{x_1}^{b} F(x)\,dx = \tfrac{1}{2}$$

The point x_1, satisfying the above property, is called a median point. Hence we conclude that x-coordinate of the optimal location of the new facility is at the median of the marginal density function with respect to x. Similar result applies to y-coordinate also.

Now it remains to determine the median of a given probability density function, say f(x). This can be done easily as described below.

Suppose f(x) is defined over $[x_1, x_2]$, i.e.,

$$\int_{x_1}^{x_2} f(x)\,dx = 1$$

Also let m be the median of f(x). Then

$$\int_{x_1}^{m} f(x)\ dx = \int_{m}^{x_2} f(x)\ dx = \tfrac{1}{2} \qquad (3.18)$$

Let $\int f(x)\ dx \equiv F(x)$

Then (3.18) can be written as

$$F(m) - F(x_1) = F(x_2) - F(m) = \tfrac{1}{2} = \tfrac{1}{2}\,[F(x_2) - F(x_1)]$$

or $\quad 2F(m) - F(x_1) - F(x_2) = 0$

This is an algebraic equation in m which can be solved, at least by a numerical procedure, to give the required median of the probability density function f(x).

Contour lines. The equation of a contour line with total cost k will be

$$k = \int_a^b \int_c^d g(x,y)\,[\,|x-x_1| + |y-y_1|\,]\ dy\ dx$$

$$= \int_a^b |x-x_1|\ F(x)\ dx + \int_c^d |y-y_1|\ G(y)\ dy$$

$$= \int_a^{x_1} (x_1-x)\ F(x)\ dx + \int_{x_1}^{b} (x-x_1) F(x) dx +$$

$$\int_c^{y_1} (y_1-y) G(y) dy + \int_{y_1}^{d} (y-y_1) G(y) dy$$

Simplifying and using

$$\int_a^b F(x)\, dx = \int_c^d G(y)\, dy = 1,$$

$$\int_a^b xF(x)\, dx = m_x = \text{mean of the marginal density function } F(x),$$

and $\displaystyle\int_c^d yG(y)\,dy = m_y$ = mean of the marginal density function $G(y)$,

we obtain the equation of the contour line as

$$k = 2\int_{x_1}^{b} xF(x)\,dx + 2\int_{y_1}^{d} yG(y)\,dy + x_1 - 2x_1\int_{x_1}^{b} F(x)\,dx + y_1 - 2y\int_{y_1}^{d} G(y)\,dy - m_x - m_y$$

The above equation can be written explicitly in terms of x_1 and y_1 if we know the marginal density functions $F(x)$ and $G(y)$, which in turn can be determined from the expression for the demand density function $g(x,y)$.

<u>3.4.2 Squared Euclidean Distance Location Problem</u> - In this case we have the following expression for the total transportation cost,

$$f(x_1,y_1) = \int_a^b \int_c^d g(x,y)\,[\,(x-x_1)^2+(y-y_1)^2\,]\,dy\,dx \qquad (3.19)$$

For getting a minimum value of $f(x_1,y_1)$, we must equate its partial derivatives with respect to x_1 and y_1 to zero.

$\partial f(x_1, y_1)/\partial x_1 = 0$ gives us

$$\int_a^b \int_c^d g(x,y) \cdot 2(x_1 - x)\, dy\, dx = 0$$

Solving for x_1, we obtain

$$x_1 = \frac{\int_a^b \int_c^d x \; g(x,y)\, dy\, dx}{\int_a^b \int_c^d g(x,y)\, dy\, dx}$$

$$= \int_a^b x\, F(x)\, dx, \text{ since denominator is equal to 1.}$$

$$= m_x$$

Similarly $y_1 = m_y$

Contour Lines. The equation of a contour line with total transportation cost k will be

$$k = \int_a^b \int_c^d g(x,y) \left[(x-x_1)^2 + (y-y_1)^2 \right] dy dx$$

$$= \int_a^b \int_c^d g(x,y)\, (x_1^2 + x^2 - 2xx_1 + y_1^2 + y^2 - 2yy_1)\, dy\, dx$$

$$= x_1^2 + m_{x_2} - 2x_1\, m_x + y_1^2 + m_{y_2} - 2y_1\, m_y$$

where m_{x_2} and m_{y_2} are respectively the second moments of F(x) and G(y).

Hence, we conclude that the contour lines for this problem are concentric circles with centre (m_x, m_y) and radius $(m_x^2 + m_y^2 - m_{x_2} - m_{y_2} + k)^{\frac{1}{2}}$.

<u>3.4.3. Euclidean Distance Location Problem.</u> When Euclidean distances are used, the expression for total cost will be

$$f(x_1, y_1) = \int_a^b \int_c^d g(x,y)\left[(x-x_1)^2 + (y-y_1)^2\right]^{\frac{1}{2}} dydx$$

For an optimal value of x_1 we must have

$$\partial f_1(x_1, y_1)/\partial x_1 = 0$$

Differentiating and solving for x_1, we obtain

$$x_1 = \frac{\displaystyle\int_a^b \int_c^d \frac{xg(x,y)}{\left[(x-x_1)^2 + (y-y_1)^2\right]^{\frac{1}{2}}}}{\displaystyle\int_a^b \int_c^d \frac{g(x,y)}{\left[(x-x_1)^2 + (y-y_1)^2\right]^{\frac{1}{2}}} dydx}$$

Hence x_1 will have to be found out iteratively, i.e., first we give approximate values to x_1 and y_1 and then improve upon the value using the above equation. Similar iterative procedure is to be used for determining optimal value of y_1. The stopping criterion and starting point similar to the previous approach may be used here also.

The use of nonlinear programming is also possible for solving the Euclidean distance location problem, but it will not be as efficient as the above iterative procedure.

Contour lines, once again, cannot be drawn directly. An indirect approach discussed previously is to be used.

Chapter 4

MULTI-FACILITY LOCATION PROBLEMS

In our previous discussion of facility location problems we treated the case of a single new facility to be located relative to an area having continuous demand for the facility. In this chapter, the previous analysis is extended to include the problem of locating multiple new facilities with respect to an area which has continuous demands for each of the new facilities. Thus the single facility location problem can be considered to be a special case of the multi-facility location problem treated here.

4.1 Discrete Demand Multi-Facility Location Problem :

First, we shall give a brief account of discrete multi-facility location problem and then switch over to its continuous version. The discrete multi-facility location problem can be formulated as follows. Let m existing facilities be located at known distinct points $P_1, \ldots, P_m$ and let n new facilities be located at points $X_1, \ldots, X_n$ in the plane. Let $d(X_j, P_i)$ represent the distance between the locations of new facility j and existing facility i and $d(X_j, X_k)$ be the distance between the locations of the new facility j and k. Let the annual cost per unit distance between new facility j and existing facility i be denoted by w_{ji}, with v_{jk} being the corresponding annual cost per unit distance between new facilities j and k. The total

annual transportation cost associated with new facilities located at $X_1, \ldots, X_n$ is given by

$$f(X_1,\ldots,X_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\, d(X_j, X_k) + \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ji}\, d(X_j, P_i) \tag{4.1}$$

because in order to evaluate the annual cost due to item movement between new facilities, it is only necessary to sum over those values of j which are less than k and over values of k from 2 to n.

Somewhat less geometrical insight is available for multi-facility location problems as compared to the single facility problems. The construction of contour lines is no longer possible, because n being at least two, the number of variables increases.

It is the costs of materials handling between the new facilities which distinguish the multi-facility location problem from the single facility problem. This follows, because when all v_{jk}'s are zero, (4.1) may be written as

$$f(X_1, \ldots, X_n) = \sum_{j=1}^{n} f_j(X_j) \tag{4.2}$$

where $$f_j(X_j) = \sum_{i=1}^{m} w_{ji}\, d(X_j, P_i) \tag{4.3}$$

We observe that (4.3) just defines a single facility total cost expression while (4.2) is the sum of n different single facility cost expressions. Since the location of one new facility has no effect upon the cost of locating other new facilities, we have

$$\min f(X_1, \ldots, X_n) = \sum_{j=1}^{n} \min f_j(X_j)$$

That is, least-cost locations of the new facility may be found by solving n single facility location problems independently. Thus, the terms v_{jk} in (4.1) give the multi-facility location problem its special character.

The new facilities j and k are said to have an exchange when v_{jk} is non-zero, and to have no exchange when v_{jk} is zero. It will always be assumed subsequently that each new facility has an exchange with at-least one other new facility.

Not only shall we assume that there is an exchange between new facilities, but also it will be assumed that there is an exchange between new and existing facilities. As a motivation for this assumption, consider a situation in which there exists a collection of new facilities that have exchanges only among those new facilities within the collection. Where should facilities within the collection be located? Obviously, all the facilities must be located at the same place and this place can be anywhere in the

region. So there is no location problem. In subsequent discussion we shall exclude such meaningless situations.

Similar to the single facility location problems, in this case also, we must take $d(X_j,X_k)$ and $d(X_j,P_i)$ as appropriately defined distances. The three types of distances discussed previously will be dealt with in this chapter also in formulating the continuous multi-facility location problems.

The discrete version of the multifacility location problem has been treated by Cabot, Francis, and Stary (3), Bindschedler and Moore (2), Francis (14), Wesolowsky and Love (44), and Pritsker and Ghare (34) for rectilinear distance; by Eyster and White (9) and White (45) for Squared Euclidean distance; and by Francis and Cabot (15), Love (26), and Miehle (30) for Euclidean distance. The methods of Eyster, White, and Wierwille (8) and Wesolowsky and Love (42) can be used for rectilinear problem as well as Euclidean problem.

4.2 Continuous Demand Multi-Facility Location Problem :

Just as we extended the discrete location problem to continuous problem for the case of single new facility, we shall consider similar extension of discrete multi-facility location problem to continuous multi-facility location problem. We shall assume that all the new facilities have demands arising all over in the region

under consideration. For solving the continuous version of the multi-facility location problem, we shall adopt two approaches, similar to those used for solving continuous single facility problems. Incidentally, the objective functions formulated by these two approaches can be minimized using the same methods for all the three types of distances. So we shall give the methods of solution for the first approach only.

4.3 Rectangular Area Approach :

In this approach we divide the region under consideration into several rectangular areas having constant demands. It is to be noted that the region will have to be divided in as many ways as there are the new facilities, i.e., we shall have to divide the region in n ways, one for each facility.

Let p_j be the total number of rectangular areas in which the region has been divided for the demand of jth new facility and u_{jk} be the constant demand for jth facility arising in kth rectangular area, $k = 1,\ldots, p_j$. Let us call this area A_{jk}. Then the total transportation cost may be written as

$$f(X_1,\ldots,X_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} u_{jk}\, d(X_j, X_k) + \sum_{j=1}^{n} \sum_{k=1}^{p_j} \iint_{A_{jk}} d(X_j, X)\, dx\, dy \qquad (4.4)$$

where X is any point in the plane.

4.3.1 Rectilinear Distance Problem - In this case (4.4) reduces to

$$f(x_1,\ldots,x_n;\ y_1,\ldots,y_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}[|x_j-x_k| + |y_j-y_k|] + \sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} [\,|x_j - x| + |y_j-y|\,]dx\ dy \qquad (4.5)$$

Since the above expression is separable in x_j's and y_j's, we can find their values independently.

(4.5) may be rewritten as

$$f(x_1,\ldots,x_n;\ y_1,\ldots,y_n) = f_1\ (x_1,\ldots,x_n) + f_2(y_1,\ldots,y_n)$$

where

$$f_1(x_1,\ldots,x_n) = \sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} |x_j-x|\ dx\ dy + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\ |x_j-x_k| \qquad (4.6)$$

and

$$f_2(y_1,\ldots,y_n) = \sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} |y_j - y|\, dx\, dy \;+$$

$$\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\, |x_j - x_k| \qquad (4.7)$$

We shall discuss the minimization of (4.6) only, since the minimization of (4.7) is an entirely similar procedure.

Let $[a_{jk}, b_{jk}]$ and $[c_{jk}, d_{jk}]$ be the boundaries of the rectangular area A_{jk} in x and y directions respectively. Then (4.6) reduces to

$$f_1(x_1,\ldots,x_n) = \sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk}\,(d_{jk} - c_{jk}) \int_{a_{jk}}^{b_{jk}} |x_j - x|\, dx \;+$$

$$\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\, |x_j - x_k|$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{p_j} U_{jk} \int_{a_{jk}}^{b_{jk}} |x_j - x|\, dx \;+$$

$$\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\, |x_j - x_k|$$

where $U_{jk} = u_{jk}\,(d_{jk} - c_{jk})$ (4.8)

The minimization of (4.8) can be easily done by the univariate method or some other technique of non-linear programming. Since the objective function is convex, the convergence will be towards the global minimum.

However, if we can tolerate some error, an approximation to (4.8) can be minimized directly by adopting the procedure given below. In this procedure, the terms $|x_j - x_k|$ and $|x_j - x|$ are taken as $[(x_j-x_k)^2 + \varepsilon]^{\frac{1}{2}}$ and $[(x_j - x) + \varepsilon]^{\frac{1}{2}}$ respectively, ε being a positive valued small constant.

We put the above mentioned approximations in (4.8), and then differentiate it partially with respect to x_j ; $j = 1,\ldots,n$; obtaining

$$\partial f_1/\partial x_j = \sum_{\substack{k=1\\k\neq j}}^{n} v_{jk}\,[(x_j-x_k)^2+\varepsilon]^{-\frac{1}{2}}(x_j-x_k) + \sum_{k=1}^{p_j} U_{jk}\int_{a_{jk}}^{b_{jk}} [(x_j-x)^2+\varepsilon]^{-\frac{1}{2}}(x_j-x)\,dx, \quad j=1,\ldots,n$$

When this is equated to zero, for getting a minimum value of f_1, we have

$$x_j = \frac{\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} v_{jk}\,x_k/[(x_j-x_k)^2+\varepsilon]^{\frac{1}{2}} + \sum_{k=1}^{p_j} U_{jk}\int_{a_{jk}}^{b_{jk}} x\,dx/[(x_j-x)^2+\varepsilon]^{\frac{1}{2}}}{\displaystyle\sum_{\substack{k=1\\k\neq j}}^{n} v_{jk}/[(x_j-x_k)^2+\varepsilon]^{\frac{1}{2}} + \sum_{k=1}^{p_j} U_{jk}\int_{a_{jk}}^{b_{jk}} dx/[(x_j-x)^2+\varepsilon]^{\frac{1}{2}}},$$

$$j=1,\ldots,n$$

The above expression can be used as an iterative formula for obtaining x_j's. Similar formula can be derived for y_j's.

4.3.2 Squared Euclidean Distance Problem - In this case the total cost will be given by

$$f(x_1,\dots x_n;y_1,\dots,y_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}[(x_j-x_k)^2+(y_j-y_k)^2]+$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} [(x_j-x)^2+(y_j-y)^2]\,dxdy$$

Since this expression is separable, we can independently find out the x and y coordinates of the new facilities. x co-ordinates can be found out by the minimization of

$$f_1(x_1,\dots,x_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}(x_j-x_k)^2 \quad +$$

$$\sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} (x_j-x)^2\,dx\,dy \qquad (4.10)$$

For a minimum value of f_1, we must have

$$\partial f_1/\partial x_j = 0 \quad ; \quad j = 1,\dots,n$$

$$r \quad \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} 2\,(x_j - x)\,dxdy + \sum_{\substack{k=1\\k\neq j}}^{n} 2\,v_{jk}(x_j-x_k)=0$$

Or,

$$x_j = \frac{\sum_{k=1}^{p_j} \iint_{A_{jk}} x \, dx \, dy + \sum_{\substack{k=1 \\ k \neq j}}^{n} v_{jk} x_k}{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{kj}} dx \, dy + \sum_{\substack{k=1 \\ k \neq j}}^{n} v_{jk}}, \quad j = 1, \ldots, n.$$

Similar expressions for y_j's can be determined by using the same arguments. Hence we observe that we shall have to solve two systems of simultaneous linear equations, each system having n equations.

4.3.3 Euclidean Distance Problem - The total transportation cost of the Euclidean distance problem is

$$f(x_1, \ldots, x_n; y_1, \ldots, y_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk} \left[(x_j - x_k)^2 + (y_j - y_k)^2 \right]^{1/2}$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \left[(x_j - x)^2 + (y_j - y)^2 \right]^{\frac{1}{2}} dx dy$$

We observe that our objective function is a non-linear convex function of the independent variables, x_j and y_j, $j=1,\ldots,n$, so non-linear programming techniques can be successfully used to obtain a convergent global optimal solution to the problem.

$= [(x_j-x_k)^2 + (y_j-y_k)^2]$

Alternately we can solve this problem in the following way also. We know that for a minimum value of f, we must have

$$\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial y_j} = 0, \quad j = 1,\ldots,n$$

or

$$\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{(x_j-x)}{[(x_j-x)^2+(y_j-y)^2]^{\frac{1}{2}}} dx\, dy + \sum_{\substack{k=1\\k\neq j}}^{n} v_{jk} \frac{(x_j-x_k)}{[(x_j-x_k)^2 + (y_j-y_k)^2]^{\frac{1}{2}}}$$

$$\text{or } x_j = \frac{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{x\,dxdy}{E_j} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}\, x_k}{D_{jk}}}{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{dxdy}{E_j} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}}{D_{jk}}} \tag{4.11}$$

$$\text{Similarly } y_j = \frac{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{y\,dxdy}{E_j} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}\, y_k}{D_{jk}}}{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{dxdy}{E_j} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}}{D_{jk}}} \tag{4.12}$$

where $E_j = [(x_j-x)^2 + (y_j-y)^2]^{\frac{1}{2}}$

and $D_{jk} = [(x_j-x_k)^2 + (y_j-y_k)^2]^{\frac{1}{2}}$

If, for any j and k, $(x_j,y_j) = (x_k, y_k)$, then D_{jk} becomes zero and (4.11) and (4.12) are undefined. Consequently, we see that difficulties arise when the locations for any two new facilities coincide (mathematically) with each other. If there were some guarantee that the optimum locations of any two new facilities would never be the same, then (4.11) and (4.12) would still give the solution to our problem, because these equations, then, could be solved iteratively to give the optimal locations of new facilities. Unfortunately, there is no such guarantee available. Consequently, a modification of this approach is required. The proposed modification is to use an approximation of D_{jk} instead of D_{jk} in (4.11) and (4.12). This approximation is to be such that even if for some j and k, $(x_j, y_j) = (x_k, y_k)$, D_{jk} does not equal zero. The proposed approximation is to use

$$D'_{jk} = [\, (x_j-x_k)^2 + (y_j-y_k)^2 + \varepsilon \,]^{\frac{1}{2}}$$

in place of D_{jk}, where ε is an arbitrarily small, positive-valued constant. The value of ε can be determined by a compromise between how quickly we want to get the solution of our problem and the accuracy of that solution.

The final iteration equations can be given by

$$x_j^{(h+1)} = \frac{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{x\,dx\,dy}{E_j^{(h)}} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}\, x_k^{(h)}}{D'^{(h)}_{jk}}}{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{dx\,dy}{E_j^{(h)}} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}}{D'^{(h)}_{jk}}}$$

$$y_j^{(h+1)} = \frac{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{y\,dx\,dy}{E_j^{(h)}} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}\, y_k^{(h)}}{D'^{(h)}_{jk}}}{\sum_{k=1}^{p_j} u_{jk} \iint_{A_{jk}} \frac{dx\,dy}{E_j^{(h)}} + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{v_{jk}}{D'^{(h)}_{jk}}}$$

4.4 Density Function Approach :

In this approach we define the demand density functions $f_j(x,y)$, $j=1,\ldots,n$, over the entire region under consideration for all the new facilities to be located. Then the total cost of transportation will be given by

$$f(X_1,\ldots,X_n) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} v_{jk}\, d(X_j,X_k) + \sum_{j=1}^{n} \iint_A f_j(x,y)\, d(X_j,X)\,dx\,dy$$

where A is the area under consideration, and $d(X_j,X_k)$ and $d(X_j,X)$ denote appropriately defined distances.

4.4.1 Rectilinear Distance Problem - The total cost function, in this case, becomes

$$f(x_1,\ldots,x_n;\ y_1,\ldots,y_n) = \sum_{j=1}^{n-1}\ \sum_{k=j+1}^{n} v_{jk}[\,|x_j-x_k| + |y_j-y_k|\,] + \sum_{j=1}^{n} \iint_A f_j(x,y)\,[\,|x_j-x| + |y_j-y|\,]\,dx\,dy$$

which is separable in x_j's and y_j's. The cost component corresponding to x_j's is

$$f_1(x_1,\ldots,x_n) = \sum_{j=1}^{n-1}\ \sum_{k=j+1}^{n} v_{jk}\,|x_j-x_k| + \sum_{j=1}^{n} \iint_A f_j(x,y)\,|x_j-x|\,dx\,dy$$

which can be either minimized by the univariate method of nonlinear programming or by the procedure given in section 4.3.1.

<u>4.4.2 Squared Euclidean Distance Problem</u> - In the case of squared Euclidean distance the expression for total transportation cost will be given by

$$f(x_1,\ldots,x_n;\ y_1,\ldots,y_n) = \sum_{j=1}^{n-1}\ \sum_{k=j+1}^{n} v_{jk}\,[(x_j-x_k)^2 + (y_j-y_k)^2] + \sum_{j=1}^{n} \iint_A f_j(x,y)\,[(x_j-x)^2 + (y_j-y)^2]\,dx\,dy$$

This function can be minimized in a manner similar to one given in section 4.3.2.

4.4.3 Euclidean Distance Problem - The objective function for the Euclidean distance problem is given by

$$f(x_1,\dots,x_n;\ y_1,\dots,y_n) = \sum_{j=1}^{n-1}\ \sum_{k=j+1}^{n} v_{jk}\left[(x_j-x_k)^2 + (y_j-y_k)^2\right]^{\frac{1}{2}} + \sum_{j=1}^{n} \iint_A f_j(x,y)\left[(x_j-x)^2+(y_j-y)^2\right]^{\frac{1}{2}} dx\,dy$$

This function is also convex and amenable to solve by any non-linear programming technique which could be used for unconstrained problems. But a close approximation to the above objective function can be minimized by a procedure similar to that of 4.3.3.

Chapter 5

A PROBABILISTIC FACILITY LOCATION PROBLEM

The problem which we shall deal in this chapter is a special type of discrete demand single facility location problem,, rather than a continuous demand location problem. We, still, propose to discuss it here because it is very much similar to the continuous demand single facility location problem as far as its nature and method of solution are concerned. Furthermore, it is an important problem which arises in real life quite frequently and is an extension of the discrete demand single facility location problem.

All the previous research work on the discrete demand facility location problems assumes that the locations of the destinations or the existing facilities are deterministic. In the present formulation, the destinations are no longer predetermined points, but random variables with given probability distributions. The problem we now wish to solve is to find the location of a point, the sum of the expected values of whose weighted distances from all destination points is a minimum. This model is useful in all real life situations where some uncertainty is associated with the location of the destinations.

In order to formulate the problem, let us assume that we are given n fixed destinations; $P_1, \ldots, P_n$; whose co-ordinates are (x_j, y_j), $j = 1, \ldots, n$, and each P_j has associated with it a probability density function $f_j(x_j, y_j)$. The demand at jth destination, i.e., at P_j is assumed to be w_j and the new facility is located at a point X whose co-ordinates are (x, y). Then the expected value of the weighted distance between new facility and jth destination, i.e., P_j will be given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j \, d(X, P_j) \, f_j(x_j, y_j) \, dx_j \, dy_j$$

and hence our objective function considering all destinations will be

$$f(X) = \sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j \, d(X, P_j) \, f_j(x_j, y_j) \, dx_j \, dy_j \qquad (5.1)$$

which is to be minimized with respect to x and y.

We observe that the form of f(x, y) is similar to the objective function for continuous demand single facility location problem formulated by the density function approach, with the only exception that in (5.1) one additional summation sign appears in the beginning. But, fortunately, this difference does not alter the situation much and the solution procedures used for continuous demand single facility problem can be adopted with slight modifications.

The modifications for all the three distances are presented in the following paragraphs.

1. Rectilinear Distance Problem. When rectilinear distances are used (5.1) takes the form

$$f(X) = f_1(x) + f_2(y)$$

$$\text{where } f_1(x) = \sum_{j=1}^{n} w_j \int_{-\infty}^{\infty} |x_j - x| \; F_j(x_j) \, dx_j$$

$$f_2(y) = \sum_{j=1}^{n} w_j \int_{-\infty}^{\infty} |y_j - y| \; G_j(y_j) \, dy_j$$

and $F_j(x_j)$ and $G_j(y_j)$ are marginal density functions of $f_j(x_j, y_j)$ with respect to x_j and y_j respectively. The optimal values of x and y can be obtained by the independent minimization of $f_1(x)$ and $f_2(y)$.

By arguments similar to those used to solve continuous demand single facility location problem by density function approach, we obtain following condition of optimality for x. Similar condition can be had for y also.

$$\sum_{j=1}^{n} w_j \left[H_j(\infty) + H_j(-\infty) - 2H_j(x) \right] = 0 \tag{5.2}$$

where $H_j(x_j) \equiv \int F_j(x_j) dx_j, \quad j = 1, \ldots, n$

5.2) is an algebric equation in x and can be solved (at least numerically) to yield an optimal value of it.

2. Squared Euclidean Problem. Proceeding exactly on the same lines as we did in Chapter 2 with density function approach, we can obtain the following optimal values of x and y.

$$x = \sum_{j=1}^{n} w_j \, m_{x_j} \Big/ \sum_{j=1}^{n} w_j$$

and $$y = \sum_{j=1}^{n} w_j \, m_{y_j} \Big/ \sum_{j=1}^{n} w_j$$

where m_{x_j} and m_{y_j} are respectively the means of the marginal density functions $F_j(x_j)$ and $G_j(y_j)$. This shows that the probabilistic problem for squared Euclidean distance can be solved by assuming it to be a deterministic one, with the co-ordinates of the existing facilities taken as the mean values of the probability density functions associated with each destination.

3. Euclidean Distance Problem. When Euclidean distances apply, we can devise an iterative scheme similar to the one used for continuous demand single facility location problem. The iteration formulae will be

$$x^{(h+1)} = \frac{\sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j \, x_j f_j(x_j, y_j)\, dx_j \, dy_j / d_j^{(h)}}{\sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j f_j(x_j, y_j)\, dx_j dy_j / d_j^{(h)}}$$

$$\text{and } y^{(h+1)} = \frac{\sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j y_j f_j(x_j, y_j)\, dx_j\, dy_j / d_j^{(h)}}{\sum_{j=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_j f_j(x_j, y_j)\, dx_j\, dy_j / d_j^{(h)}}$$

$$\text{where } d_j^{(h)} = \left[\left\{ x_j - x^{(h)} \right\}^2 + \left\{ y_j - y^{(h)} \right\}^2 \right]^{1/2}$$

and superscript denotes the iteration number.

single and multifacility location problems for all the three types of distances have been treated by this approach. It is to be pointed out that this approach is easier and more accurate because dividing a region into rectangular areas with some demand is a difficult task, involves quite a sufficient amount of error, and no systematic procedure is available to do it. On the other hand, the determination of a demand density function can be done by evaluating the demand at points of a grid constructed over the region, and then by interpolating between them, which is a systematic procedure easily done by the techniques of numerical analysis. The accuracy of this procedure can also be judged. Furthermore, the solution procedures of highly similar nature are obtained in both of the two approaches, especially for squared Euclidean and Euclidean problems, hence it is better to use density function approach.

A probabilistic model of the discrete demand single facility location problem has also been developed. In this model the locations of destinations or existing facilities are random variables with known probability distribution functions. It turns out that this problem can be solved exactly in the same manner as the continuous demand single facility location problem is solved by the

The procedure for developing the contour lines for these probabilistic models has not been discussed but it is hoped that this can be done easily. Further the methodology adopted for the probabilistic model of the discrete location problem can be extended to the multi-facility location problems.

Some continuous demand facility location problems arising in practice may have some constraints upon the location of the new facilities. If these constraints exclude some part of the region, still making some continuous space available for locating the new facilities, then the problems can be formulated as in this thesis (of course, they will have to be solved by non-linear programming; the special methods devised will not work), but if the constraints are such that they leave only some discrete points available for locating the new facilities, then new methods will be required to handle the problem. Therefore, further research efforts need to be directed towards the extension of the discrete space location problems discussed in first chapter for continuous demand situations.

Another scope for future research is to extend the continuous demand multifacility location problem to one involving similar facilities. Both uncapacitated and capacitated versions of the similar facility problem may be undertaken. One can also extend the models presented in this work to those which involve both discrete and continuous (point and area) destinations, or to those involving a combination of various distances because some facilities may be connected by a straight line distance whereas others, by rectangular aisles. The location theory models developed in this thesis pertain to two dimensional space. Attempts should be made to extend these models to three dimensions to represent a situation in which facilities are to be located in a multi-storeyed building at different floors.

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